Fermion Decoupling and the Axial Anomaly on the Lattice

H. Banerjee[a] and Asit K. De[b*]

[a]S.N.Bose National Centre for Basic Sciences, JD Block, Salt Lake, Calcutta 700 091, India

[b]Saha Institute of Nuclear Physics, 1/AF Salt Lake, Calcutta 700064, India

In the axial Ward identity of lattice QED we show that in the limit of infinite fermion mass m the pseudoscalar density term exactly cancels the Adler-Bell-Jackiw (ABJ) anomaly. Using this result we calculate the U(1) axial anomaly in a non-abelian gauge theory.

1. Introduction.

Wilson fermions break chiral symmetry explicitly. Explicit breaking of chiral symmetry is necessary to generate, from the lattice regulated model, the U(1) axial anomaly in the continuum limit for a vectorlike gauge theory [1].

To examine the role of the underlying lattice fermion model in generating the ABJ anomaly a convenient and transparent starting point is the condition, in this context, for decoupling of the fermion in the large mass limit from the background gauge field [2],

$$\begin{aligned}\langle \Delta_\mu J_{\mu 5}(x)\rangle_{a=0} &= 2im\ \langle\overline{\psi}_x\gamma_5\psi_x\rangle_{a=0}\\ &- \lim_{m\to\infty}\left[2im\langle\overline{\psi}_x\gamma_5\psi_x\rangle_{a=0}\right] \quad (1)\end{aligned}$$

where a is the lattice constant. One recognises Eq.(1) as the Adler condition [3] which states that the triangle graph amplitude should vanish in the limit as the mass of the loop fermion becomes infinite. To establish that the decoupling condition is indeed equivalent to the axial Ward identity one needs the supplementary relation

$$\lim_{m\to\infty}\left[2im\langle\overline{\psi}_x\gamma_5\psi_x\rangle_{a=0}\right] = \frac{ig^2}{16\pi^2}\epsilon_{\mu\nu\lambda\rho}\mathrm{tr}F_{\mu\nu}F_{\lambda\rho} \quad (2)$$

where $F_{\mu\nu}$ is the gauge field tensor.

Our derivation of Eqs.(1) and (2) in lattice QED demonstrates that as long as the underlying lattice fermion model removes doubling completely and is gauge-invariant and local, the ABJ anomaly is generated without reference to the specific form of the irrelevant term. In non-abelian gauge theories on lattice Eq.(1) provides, as we shall see, a simple recipe for deriving the U(1) axial anomaly.

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2. Decoupling in QED.

The key to our analysis is the Rosenberg [4] tensor decomposition of the amplitude of the triangle diagrams (i) and (ii) in continuum QED for axial current $j_{\lambda 5}(x)$ to emit two photons with momenta and polarisation (p,μ) and (k,ν):

$$\begin{aligned}T^{(i+ii)}_{\lambda\mu\nu} &= \epsilon_{\lambda\mu\nu\alpha}k_\alpha A(p,k,m)\\ &+ \epsilon_{\lambda\nu\alpha\beta}p_\alpha k_\beta[p_\mu B(p,k,m)+k_\mu C(p,k,m)]\\ &+ [(k,\nu)\leftrightarrow(p,\mu)]. \quad (3)\end{aligned}$$

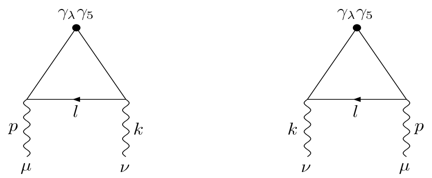
Gauge invariance relates the Rosenberg form factors B and C to A.

$$A(p,k,m) = p^2 B(p,k,m) + p.k\, C(p,k,m). \quad (4)$$

The form factors B and C are of mass dimension -2, and, therefore, must vanish as m^{-2} for large fermion mass. Gauge invariance then guarantees that

$$\begin{aligned}&\lim_{m\to\infty}(p+k)_\lambda T^{(i+j)}_{\lambda\mu\nu} =\\ &-\epsilon_{\mu\nu\alpha\beta}p_\alpha k_\beta \lim_{m\to\infty}[A(p,k,m)+A(k,p,m)] \quad (5)\\ &= 0, \quad (6)\end{aligned}$$

which is the basis of Eq.(1). In the above, (6) follows from (5) because of (4) and the asymptotic behavior of B and C.



(i) (ii)

On lattice, the decoupling condition (6) should be realised in the continuum limit irrespective of the underlying model for fermion as long as it is free from doublers and local. The form factors B and C which are highly convergent amplitudes must coincide with their respective expressions in the continuum in all *legitimate* lattice models. Residual model dependence, if any, can appear only in the form factor A because of potential logarithmic divergence. This, however, is ruled out by the gauge invariance constraint (4).

In lattice QED with Wilson fermions, the Feynman amplitudes corresponding to the two diagrams (i) and (ii) are :

$$[T^{(i+ii)}_{\lambda\mu\nu}]_a = -g^2\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 l}{(2\pi)^4} \mathrm{Tr}\Big[\gamma_\lambda\gamma_5 \cos a(l+\frac{k-p}{2})_\lambda \; S(l-p)V_\mu(l-p,l)S(l) \; V_\nu(l,l+k)S(l+k) \; +(k,\nu\leftrightarrow p,\mu)\Big], \quad (7)$$

with the fermion propagator $S(l)$ and the one-photon vertex $V_\mu(p,q)$ given by

$$S(l) = \left[\sum_\mu \gamma_\mu \frac{\sin a l_\mu}{a} + \frac{r}{a}\sum_\mu (1-\cos a l_\mu) + m\right]^{-1}$$

$$V_\mu = \gamma_\mu \cos\frac{a}{2}(p+q)_\mu + r\sin\frac{a}{2}(p+q)_\mu .$$

where r is the Wilson parameter. *We have the same γ-matrix convention as in* [1]. On lattice there are four additional diagrams with *irrelevant* vertices. As will be evident from the following, they do not contribute in the continuum limit.

The lattice amplitude (7) is superficially linearly divergent. However, the leading term, obtained by setting the external momenta $p,\ k=0$ is odd in the loop momentum l and vanishes due to symmetric integration. The amplitude, therfore, vanishes at least linearly in external momenta as indeed the Rosenberg decomposition suggests and, furthermore, the effective divergence is at most logarithmic.

Our strategy is to consider the derivative of (7) with respect to the fermion mass m rather than the external momenta $p,\ k$ as is common practice

$$[R^{(i)}_{\lambda\mu\nu}]_a \equiv \frac{d}{dm}[T^{(i)}_{\lambda\mu\nu}]_a. \quad (8)$$

Lattice power counting gives a negative integer for the effective degree of divergence of $[R^{(i)}_{\lambda\mu\nu}]_a$. One can, therefore, take, thanks to the Reisz theorem [5], the continuuum limit of the integrands and evaluate the loop integrals in the entire phase space $-\infty \le l_\mu \le \infty$ as in the continuum. In the continuum limit, amplitudes of only two diagrams (i) and (ii) survive and amplitudes with *irrelevant* vertices vanish. The amplitudes $[R^{(i)}_{\lambda\mu\nu}]_{a=0}$ and $[R^{(ii)}_{\lambda\mu\nu}]_{a=0}$ are individually Bose-symmetric and hence gauge-invariant

The Rosenberg tensor decomposition is
$[R^{(i+ii)}_{\lambda\mu\nu}]_{a=0} =$

$$4g^2 m\int_{-\infty}^{\infty}\frac{d^4 l}{(2\pi)^4}\Big[\mathrm{Tr}(\gamma_5\gamma_\lambda\gamma_\mu\gamma_\nu \not{p}) \left(\frac{1}{D}(1+\frac{k^2}{d_3}) - \frac{1}{d_1 d_3^2}\right) + \mathrm{Tr}(\gamma_5\gamma_\lambda\gamma_\nu\not{p}\not{k})\frac{2(l_\mu - p_\mu)}{D d_1} + (p,\mu\leftrightarrow k,\nu)\Big] \quad (9)$$

where $D = d_1 d_2 d_3$ and
$d_1 \equiv (l-p)^2 + m^2,\ d_2 \equiv l^2 + m^2,\ d_3 \equiv (l+k)^2 + m^2$.

The four-divergence of the amplitude for the axial vector current is to be obtained from

$$[(p+k)_\lambda R^{(i+ii)}_{\lambda\mu\nu}]_{a=0} = \frac{d}{dm}[(p+k)_\lambda T^{(i+ii)}_{\lambda\mu\nu}]_{a=0} \quad (10)$$

$$= -\frac{1}{\pi^2}\epsilon_{\mu\nu\alpha\beta}p_\alpha k_\beta \frac{d}{dm}\int_{0\le s+t\le 1}\frac{m^2}{c^2+m^2}ds\ dt, \quad (11)$$

with $c^2 \equiv s(1-s)p^2 + t(1-t)k^2 + 2st\ p.k.$ (12)

The Adler condition (1) determines the *constant of integration*

$$[(p+k)_\lambda T^{(i+ii)}_{\lambda\mu\nu}]_{a=0} =$$
$$-\frac{1}{\pi^2}\epsilon_{\mu\nu\alpha\beta}p_\alpha k_\beta \left[\int \frac{m^2}{c^2+m^2} ds\ dt - \frac{1}{2}\right]. \qquad (13)$$

The ABJ anomaly is identified as the $m = 0$ limit of the right hand side of (13):

$$\text{ABJ anomaly} = \frac{1}{2\pi^2}\epsilon_{\mu\nu\alpha\beta}p_\alpha k_\beta \qquad (14)$$

3. U(1) axial anomaly in non-abelian gauge theories.

The representation motivated by the decoupling condition (1):

$$\lim_{m\to\infty}\left[2im\langle\overline{\psi}_x\gamma_5\psi_x\rangle_{a=0}\right] =$$
$$\lim_{m\to\infty}\left[2im\langle x|\mathrm{Tr}\gamma_5(\not{D}+W+m)^{-1)}|x\rangle_{a=0}\right] \qquad (15)$$

constitutes the starting point of our calculation of the axial anomaly in non-Abelian theories, *e.g.*, lattice QCD. The Dirac operator $\not{D}$ and the Wilson term W are given by

$$\begin{aligned} D_\lambda &\equiv \frac{1}{2ia}\left(e^{ip_\lambda a}U_\lambda - U^\dagger_\lambda e^{-ip_\lambda a}\right) \qquad (16)\\ W &\equiv \frac{r}{2a}\sum_\lambda\left(2 - e^{ip\lambda a}U_\lambda - U^\dagger_\lambda e^{-ip_\lambda a}\right) \end{aligned}$$

where $U_\lambda \equiv exp(iagA_\lambda)$ is the link variable with $A_\lambda \equiv t^a A^a_\lambda$ the gauge potential and t^a the generators of $SU(N)$.

Our strategy is to develop the Green function for lattice fermion in a perturbative series:

$$\begin{aligned} &(\not{D}+W+m)^{-1} = (-\not{D}+W+m)G, \text{ with}\\ G &= \left(-\not{D}^2 + (W+m)^2 + [\not{D}, W]\right)^{-1}\\ &= G_0 - gG_0VG_0 + g^2G_0VG_0VG_0 + ... \qquad (17) \end{aligned}$$

where the *free* part $G_0 = \left[\sum \frac{\sin^2 ap_\mu}{a^2} + \left(\frac{r}{a}\sum_\mu(1-\cos ap_\mu)+m\right)^2\right]^{-1}$ is of Reisz degree -2 and has the expected continuum limit $(p^2+m^2)^{-1}$.

The potential gV has three pieces

$$gV = gV_0 + gV_1 + gV_2 \qquad (18)$$

of which the first piece gV_0 is independent of γ-matrices, has Reisz degree $+1$ and non-vanishing continuum limit. The pieces gV_1 and gV_2 contain γ-matrices and each has Reisz degree zero. The continuum limit of gV_1 vanishes

$$(gV_1)_{a=0} = [\not{D}, W]_{a=0} = 0, \qquad (19)$$

whereas,

$$(gV_2)_{a=0} = \frac{i}{2}\sigma_{\mu\nu}\left[D_\mu, D_\nu\right]_{a=0} = -\frac{i}{2}\sigma_{\mu\nu}F_{\mu\nu} \qquad (20)$$

where $F_{\mu\nu}$ is the field tensor in the continuum.

The first two terms of the perturbative series (17) do not contribute simply because they do not have enough γ-matrices to give non-vanishing Dirac trace. Reisz power counting for the second and higher order terms in (17) all give negative integers. One can now use the Reisz theorem and take the continuum limit of the integrands in all these terms. Anomaly is thus given by the term which survive in the large mass limit in the continuum

$$\begin{aligned} &-\lim_{m\to\infty}\left[2img^2\langle x|\mathrm{Tr}\gamma_5 G_0V_2G_0V_2G_0|x\rangle_{a=0}\right]\\ &= -\frac{ig^2}{16\pi^2}\epsilon_{\lambda\rho\mu\nu}\mathrm{tr}F_{\lambda\rho}(x)F_{\mu\nu}(x) \qquad (21) \end{aligned}$$

where 'tr' now denotes trace over internal symmetry indices. Note that the final result (21) is local, all nonlocalities disappearing in the large m limit, as do all higher order terms in the perturbative series (17).

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